LITERATURE CITED

1. E. I. Bichenkov, A. A. Deribas, et al., "Welding by explosion," Narodnokhozyaistvennoe Ispol'zovaniya Vzryva, No. 22 (1962).
2. V. S. Sedykh, A. A. Deribas, et al., "Welding by explosion," Svarochnoe Proizvodstvo, No. 5 (1962).
3. A. Platini, "Sulla confluenza di due vene," Atti del R. Isit. Venetto di So. L. ed Arti., 75 (1916).
4. M. A. Lavrent'ev, "Cumulative discharge and principles of its operation," Usp. Mat. Nauk, 12, No. 4(76) (1957).

ASYMPTOTIC EXPANSIONS OF THIN AXISYMMETRIC CAVITIES

A. G. Petrov

UDC 532.58.33

The theory of flow around a thin body [1] enables one to obtain expansions for the potential of the velocity field in terms of a small parameter χ (thickness of the body) with any degree of accuracy. The first six terms of this expansion have the following orders of magnitude: 1, $\chi^2 \ln\chi$, χ^2, $\chi^4 \ln^2\chi$, $\chi^4 \ln\chi$, χ^4. In most works on cavitating flows, calculations are carried out by using only the two first terms of the expansion, e.g., [2]. The problem of determining the free boundary reduces, in that approximation, to solution of an ordinary differential equation. For practical reasons one should take into account also the third term of the expansion together with the second, which is of the order close to χ^2, while the subsequent three terms of the expansion are of essentially smaller, close to χ^4, order. In presence of the term χ^2, the potential of the flow is expressed by the integral operator acting on the function defining the boundary of the body placed in the stream [1]. Therefore, the equation of the free boundary is a nonlinear integrodifferential equation. It seems that only [3] contains calculations in this approximation. The solution of the integrodifferential equation is shown in the form of an expansion in negative powers of $\ln\chi$. In this work the Riabushinskii scheme is used in order to obtain an asymptotic expansion for the drag force F in powers of a small parameter ε_1 for arbitrary thickness of the cavitating body. The first term of this expansion agrees with the asymptotic formula given in [4]. For the flow in the Kirchhoff scheme ($\sigma = 0$) an expansion is obtained for $x \to \infty$ for the free-jet boundary. Its asymptotic behavior agrees with the law of jet expansion obtained independently by Gurevich and Levinson [5].

1. Theory of Nonseparating Flow around a Thin Body

Here we consider the problem of flow around a thin body of rotation by a stationary stream of nonviscous incompressible fluid. Let all lengths be referred to the half-length of the body ℓ_x, velocities be referred to the velocity of the incoming stream at infinity v_∞, and the boundary of the body in the meridional plane be defined by the equation

$$y = \chi f(x),\ -1 \leqslant x \leqslant 1. \tag{1.1}$$

The small parameter $\chi \ll 1$ is a measure of the relative thickness of the body whose shape is given by the function f(x). The potential Φ of the velocity field is to be found from the solution of the boundary-value problem

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{1}{y}\frac{\partial\Phi}{\partial y} = 0, \tag{1.2}$$

$$\frac{\partial\Phi}{\partial y}\Big/\frac{\partial\Phi}{\partial x} = \chi\frac{df}{dx},\quad |x| \leqslant 1,\quad y = \chi f(x),$$

$$\Phi \to x,\ x^2 + y^2 \to \infty.$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 45-49, September-October, 1986. Original article submitted June 17, 1985.

0021-8944/86/2705-0667$12.50

The analytical method of matching asymptotic expansions for the solution of (1.2) is shown in [1]. Following that method, it is possible to obtain an expansion for the potential Φ in a neighborhood of the thin body

$$\Phi = x + L(z') + 2z' \ln \frac{y}{2} + O(\chi^4 \ln^2 \chi), \tag{1.3}$$

where the function z(x) and the linear operator L are determined by the formulas

$$z = (1/4)\chi^2 f^2(x), \; L(z) = -\ln(1 - x^2)z(x) + I(z); \tag{1.4}$$

$$I(z) = \int_{-1}^{1} \frac{z(x) - z(y)}{|x - y|} dy. \tag{1.5}$$

The convenience of the form of the integral (1.5) is noted in [6] when the distribution of the cross-sectional surface is given by a polynomial. In this case the integrand also becomes a polynomial.

Starting with the obtained asymptotic formula for the potential (1.3) it is not difficult to determine the velocity at the boundary of the thin body:

$$\frac{v^2}{v_\infty^2} = \left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 = 1 + 2\frac{d}{dx} L(z') + 2z'' \ln z + \frac{z'^2}{z} + O(\chi^4 \ln^2 \chi). \tag{1.6}$$

If the following transformation is employed:

$$\frac{d}{dx} L(z') = \frac{z'(1)}{1-x} - \frac{z'(-1)}{1+x} - z'' \ln(1 - x^2) + I(z''),$$

the formula (1.6) will take the form

$$\frac{v^2}{v_\infty^2} = 1 + 2z'' \ln \frac{z}{1-x^2} + 2I(z'') + \frac{z'^2}{z} + 2\frac{z'(1)}{1-x} - 2\frac{z'(-1)}{1+x}. \tag{1.7}$$

The eigenfunctions of the linear integral operator I are the Legendre polynomials $P_n(x)$, and the eigenvalues are given by the formula

$$I(P_n) = \lambda_n P_n, \quad \lambda_0 = 0, \quad \lambda_n = 2\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right), \quad n \geqslant 1. \tag{1.8}$$

In this way, when the boundary of the thin body z(x) is represented by a polynomial, the velocity v is calculated from (1.7) and (1.8) generally without use of quadratures.

2. Variational Formulation of the Problem of Cavitating Flow. Force Acting on the Cavitating Body

Here we consider an axisymmetric flow, separating from the surface, around an arbitrary body according to the Riabushinskii scheme. From formula (1.6) we obtain the equation of free boundary where the condition $v = v_k$ is satisfied:

$$z'' \ln z + \frac{z'^2}{2z} + \frac{d}{dx} L(z') = \frac{\sigma}{2}, \quad \sigma = \frac{v_k^2}{v_\infty^2} - 1 \tag{2.1}$$

(σ is the cavitation number).

A variational formulation can be given to Eq. (2.1) by employing the Riabushinskii variational principle: the functional $V\sigma - M$ reaches an extremal value at the free energy (V and M are the volume and the added mass of the body) [8].

The added mass is determined from the well-known formula

$$M = -\int (\Phi - x) n_x dS, \quad n_x dS = 4\pi z' dx,$$

where the integral is taken over the surface of the axisymmetric body $y = \chi f(x)$, n_x is projection of the external normal to the surface on the x axis; dS, surface element between two cross sections at points x and x + dz. Substituting expression (1.5) for Φ in the integral, we have

$$M = -4\pi \int_{-1}^{1} (z' \ln z + L(z')) z' dx.$$

The volume of the axisymmetric body is

$$V = 4\pi \int_{-1}^{1} z dx.$$

Hence, in accordance with the Riabushinskii principle, the functional

$$U = (V\sigma - M)/4\pi = \int_{-1}^{1} (\sigma z + z'^2 \ln z + z' L(z'))\, dx \tag{2.2}$$

takes an extremal value at the free boundary.

It is not difficult to show that a solution of Eq. (2.1) is an extremum of the functional U. Equation (2.1) allows one to determine the shape of the cavity with a relative error of $\chi^2 \ln^2 \chi$. However, calculation of the force acting on the cavitating body from (2.1) by direct integration of the pressure on its surface is impossible because, in the neighborhood of the singularity at x = −1 the expansion (2.1) is meaningless, and the pressure at this point has a nonintegrable singularity.

This difficulty can be overcome in the following way. It turns out that the drag force can be expressed by the extremal value of the functional (2.2) U_0:

$$F = 3\pi\rho v_\infty^2 l_x^2 U_0 (1 + O(l/l_x)), \tag{2.3}$$

where $2\ell_x$ is the length of the cavity, and ℓ is a characteristic dimension of the cavitating body. For a cone and disc this formula, as obtained by Garabedian [4, 5], is exact. It was shown in [8] that its error for cavitating bodies of arbitrary form for $\sigma \to 0$ converges to zero like the inverse of the length of the cavity. The relative error of formula (2.3) is of the order of ℓ/ℓ_x and does not exceed the relative error of Eq. (2.1), $\chi^2 \ln^2 \chi$. In this way, in addition to the existing theories [1-3], Eq. (2.3) allows one also to determine the force acting on the cavitating body of arbitrary form with an error controlled by the asymptotic approximation (2.1).

3. Calculation of the Free Boundary by Method of Matching the Asymptotic Expansions

Following [3], it is possible to introduce two separate parameters of thinness: for the cavitating body χ_1, and for the cavity χ (for the disc $\chi_1 = \infty$). We will consider the flow around a cavitating body of arbitrary thickness $\chi/\chi_1 \to 0$ based on the Riabushinskii scheme. Dimensions of the section with the larger parameter of thinness become infinitely small in the limit, allowing one to consider the cavity as a thin body in this case as well.

External Expansion. In solving Eq. (2.1) it is convenient to introduce a small parameter ε and an unknown function ζ related to the cavitation number σ and to the function z by the formulas

$$\sigma = \varepsilon \ln(1/\varepsilon),\ z = \varepsilon\zeta. \tag{3.1}$$

Now (2.1) can be written in the form

$$\zeta'' + \frac{1}{2} = \varepsilon_1 \left[\frac{d^2}{dx^2}\left(\zeta \ln \frac{\zeta}{1-x^2} - \zeta + I(\zeta)\right) - \frac{\zeta'^2}{2\zeta} \right]. \tag{3.2}$$

Without loss of accuracy of the considered approximation at the limit points, we impose the conditions

$$\zeta(1) = \zeta(-1) = 0. \tag{3.3}$$

In fact, in ε-neighborhoods of the limit points 1 and -1, the values ζ, as will be seen from the solution, appears as a small number of the order of ε (physically it means that the square of the width of the cavitating body is at least ε times smaller than the square of the cavity's midship section). Thus, the more accurate conditions $\zeta(1) \sim \varepsilon$ can be replaced by (3.3), causing an error of the order of ε in determining ζ, and an error of the order of ε^2 in determining z. That lies outside the limits of accuracy of the considered approximation.

The solution of Eq. (3.2) can be sought in the form of expansion in powers

$$\zeta = \zeta_0 + \varepsilon_1\zeta_1 + \varepsilon_1^2\zeta_1^2 + \ldots \tag{3.4}$$

All terms of expansion (3.4) are asymptotically accurate as the error of the order of ε is transcendentally small compared to the parameter ε_1 (it decreases faster than any power $\varepsilon_1{}^n$ when $\varepsilon_1 \to 0$).

Substituting (3.4) and (3.2) for ζ_0 and ζ_1, we obtain

$$\zeta_0'' + 1/2 = 0,$$

$$\zeta_1'' = \frac{d^2}{dx^2}\left(\zeta_0 \ln\frac{\zeta_0}{1-x^2} - \zeta_0 + I(\zeta_0)\right) - \frac{\zeta_0'^2}{2\zeta_0}.$$

The solution, satisfying conditions (3.3), takes the form

$$\zeta_0 = \frac{1}{4}(1-x^2); \tag{3.5}$$

$$\zeta_1 = \left(\frac{1}{4} - \frac{1}{2}\ln 2\right)(1-x^2) - \frac{1}{4}(1+x)\ln(1+x) - \tag{3.6}$$
$$- \frac{1}{4}(1-x)\ln(1-x) + \frac{1}{2}\ln 2.$$

The equation for the zero-order approximation for ζ_0 and its solution (3.5) are given in [2], and the second approximation (3.6) can be found in [3].

We will now determine the force acting on the cavitating body and also the degree of linear extension of the cavity. Substituting (3.4)-(3.6) in (2.2), we obtain an expansion in powers of ε_1 of the extremum of the functional U_0:

$$U_0 = \varepsilon^2 \ln\frac{1}{\varepsilon}\left(\frac{1}{6} + \frac{1-\ln 2}{3}\varepsilon_1 + \ldots\right). \tag{3.7}$$

Substituting (3.7) in (2.3), we find

$$F = \pi\rho v_\infty^2 l_x^2 \varepsilon^2 \ln\frac{1}{\varepsilon}\left(\frac{1}{2} + (1-\ln 2)\varepsilon_1 + \ldots\right). \tag{3.8}$$

Finally, the solution of (3.4)-(3.6) allows one to determine the ratio of the half-width ℓ_y of the cavity to the half-length ℓ_x

$$\chi^2 = l_y^2/l_x^2 = 4z(0) = \varepsilon(1 + \varepsilon_1 + \ldots). \tag{3.9}$$

It is very important that expansions (3.8), (3.9) are universal and independent of the geometric characteristics of the cavitating body. Its shape introduces into these expansions transcendentally small (as compared to ε_1) corrections.

Internal Expansion. In a small neighborhood of the point $x = -1$ it is convenient to introduce the internal variables X and Z

$$1 + x = \mu X, \quad z = \mu^2 Z. \tag{3.10}$$

Hence the cavity transforms by similarity into its image with the length $2\ell_x = 2/\mu$. The small parameter $\mu(\varepsilon) \to 0$ for $\varepsilon \to 0$. The exact relationship $\mu(\varepsilon)$ is determined from the matching conditions of both solutions.

For the variables defined in (3.10), Eq. (2.1) takes the form

$$Z''\ln Z+\frac{Z'^2}{2Z}=-\frac{d^2}{dX^2}\int\limits_0^{2l_x} Z'_Y \ln|Y-X|\operatorname{sgn}(Y-X)\,dY+\frac{1}{2}\varepsilon\ln\frac{1}{\varepsilon}.$$

Multiplying both sides of this equation by Z', and taking the limit $\varepsilon \to 0$, we obtain

$$\frac{d}{dX}(Z'^2\ln Z)=-2Z'\lim_{l_x\to\infty}\frac{d^2}{dX^2}\int\limits_0^{l_x} Z'_Y\ln|Y-X|\operatorname{sgn}(Y-X)\,dY. \tag{3.11}$$

It can be shown that the integral of Eq. (3.11) for $X \to \infty$ has the asymptotic expression

$$Z'^2\ln Z=1-\frac{9}{16(\ln X)^2}+\ldots, \tag{3.12}$$

where dots stand for terms of higher order of smallness for $X \to \infty$.

Hence, it is not difficult to find the main asymptotic behavior for Z(X) when $X \to \infty$:

$$Z'=(\ln X)^{-1/2}+\ldots,\quad Z=X(\ln X)^{-1/2}. \tag{3.13}$$

Matching Internal and External Expansions. In order to match the expansions (3.4) and (3.13) in the intermediate limit $\varepsilon \to 0$, one introduces an intermediate variable ξ, and a small parameter $\eta(\varepsilon)$ in the following way [1]:

$$1+x=\eta\xi,\ X=\eta\xi/\mu,\ \eta\to 0,\ \eta/\varepsilon\to\infty.$$

For fixed ξ, the main (in orders of magnitude of the small parameter η) terms of the external expansion (3.4)-(3.6) have the form

$$z=\varepsilon\zeta_0+\varepsilon\varepsilon_1\xi_1+\ldots=\frac{1}{2}\varepsilon\eta\xi-\frac{\varepsilon}{4\ln(1/\varepsilon)}\eta\xi\ln\eta\xi,$$

and the main terms of the internal expansion (3.13) have the form

$$z=\eta\xi\mu\left(\ln\xi\eta+\ln\frac{1}{\mu}\right)^{-1/2}=\mu\left(\ln\frac{1}{\mu}\right)^{-1/2}\eta\xi-\frac{1}{2}\mu\left(\ln\frac{1}{\mu}\right)^{-3/2}\eta\xi\ln\eta\xi.$$

From the matching conditions for the internal and external expansion, it follows that the coefficients of the terms $\eta\xi$ and $\eta\xi\ln\eta\xi$ must be equal. In this manner, the asymptotic expansions match when

$$\varepsilon=2\mu(\ln(1/\mu))^{-1/2}+\ldots \tag{3.14}$$

Asymptotic Law of Jet Expansion. It is easy to see that Eq. (3.11) is invariant under transformations of the one-parameter group defined by $x = cX$, $z = c^2Z$. From this and from formulas (3.12) and (3.13) we obtain the one-parameter series of geometrically similar asymptotic forms of the free surface, where

$$Z'^2\ln(Z/c^2)=c^2-(9/16)c^2(\ln(x/c))^{-2}+\ldots; \tag{3.15}$$

$$z=y^2/4=cx(\ln(x/c))^{-1/2}+\ldots \tag{3.16}$$

The cavity's length $1/\mu$ increases c times, so $\ell_x = c/\mu$; hence, by means of formulas (3.8) and (3.14), one can determine the limiting expression for force F when $\mu \to 0$:

$$F=\pi\rho v_\infty^2c^2\mu^{-2}\frac{1}{2}\varepsilon^2\ln(1/\varepsilon)\to 2\pi\rho v_\infty^2c^2. \tag{3.17}$$

The asymptotic rule (3.15) obtained from the theory of thin bodies, besides the Gurevich–Levinson main asymptotic character (3.17) [5, 8], allows one to determine the higher-order terms.

LITERATURE CITED

1. J. D. Cole, Perturbation Methods in Applied Mathematics, Ginn-Blaisdell (1968).

2. I. G. Nesteruk, "On the form of a thin axisymmetric nonstationary cavity," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 4 (1980).
3. G. V. Logvinovich and V. V. Serebriakov, "On methods of calculation of the shape of thin axisymmetric cavities," in: Hydromechanics, Vol. 32 [in Russian], Naukova Dumka, Kiev (1978).
4. P. R. Garabedian, "Calculation of axially symmetric cavities and jets," Pacific J. Math., 6, No. 4 (1956).
5. M. I. Gurevich, Theory of Jets in Ideal Fluids, Academic Press, New York (1965).
6. Kh. Eshli and M. Lendal, Aerodynamics of Wings and Bodies of Aircraft [in Russian], Mashinostroenie, Moscow (1969).
7. G. Birkhoff and E. H. Zarantonello, Jets, Wakes, and Cavities, Academic Press, New York (1957).
8. A. G. Petrov, "Simple variational method of calculation of planar and axisymmetric cavitating flows," Dokl. Akad. Nauk SSSR, 257, No. 6 (1981).

LARGE-AMPLITUDE SOLITARY INTERNAL WAVES IN A TWO-LAYER FLUID

N. V. Gavrilov UDC 532.539

The theoretical analysis of solitary waves at an interface separating two fluids of different densities is usually based on the Korteweg—de Vries equation [1-3], which has been derived not only on the standard assumption used in the theory of long waves, that the ratio of the fluid depth to the wavelength is small, but also with the additional assumption of relative smallness of the amplitude in comparison with the depth of the fluid. Consequently, the Korteweg—de Vries equation describes only small-amplitude solitary waves. Some experimental information on the range of validity of such modes may be found in [4]. A theoretical analysis of internal solitary waves without any constraints on their amplitude has been carried out in [5], where layers that move relative to one another are investigated in addition to layers that are at rest in the unperturbed state. Better experimental corroboration of the results of [5] has been obtained [4] for the velocity of wave propagation. The present article gives experimental data on the profiles of internal solitary waves, which are also in very good agreement with the model [5].

The waves were generated at an interface separating two layers of immiscible fluids of different densities, which were bounded below by a horizontal bottom and above by an impermeable horizontal cover plate. The principal notation and diagrams of the experimental arrangements are shown in Fig. 1a, b. Here H is the distance between the bottom and the cover plate, h_0 is the depth of the unperturbed lower layer, h is the depth of the perturbed lower layer, $\eta = h - h_0$ is the deviation of the interface from the equilibrium position, η_m is the amplitude, v is the velocity of propagation of the solitary waves, and ρ_0, $\rho < \rho_0$, u_0, and u are the densities and velocities of the lower and upper layers, respectively. A fixed xy rectangular coordinate system is used.

A rectangular duct with a working section of length 250 cm, width 18 cm, and height 6 cm (Fig. 1a) was used, as in [4], for the experimental creation of solitary waves in the case of fluids moving in the unperturbed state. The lower fluid could move with a velocity u_0 distributed uniformly along the vertical in the initial cross section, whereas only a slight circulatory motion took place in the upper layer in connection with friction at the interface. The working fluids were a dilute solution of salt (NaCl) in distilled water (ρ_0 = 1 g/cm^3) and kerosene (ρ = 0.8 g/cm^3). The waves were generated by a barrier in the form of a vertical plate set up at the exit from the duct and projecting above the bottom to a height b_1. Once a steady flow regime with depth h_0 of the lower fluid had been established, the barrier was raised smoothly to a height b_2 (for the generation of hummock-type waves) or was lowered (for the generation of crater-type waves) and was then brought back to its original position.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 49-54, September-October, 1986. Original article submitted July 11, 1985.

 0021-8944/86/2705-0672$12.50